

MATHS C

Modelling and Problem Solving

PHOENIX

SENIOR MATHS

PRACTICE QUESTIONS

BY TOPIC

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Topic 1

Groups

RULES TO REMEMBER

A set S forms a group under the operation ' $\circ$ ' if and only if all of the following are true:

- It is closed under ' $\circ$ ', i.e. the result is an element of S .
- It is associative, i.e. $(a \circ b) \circ c = a \circ (b \circ c)$
- There is only one identity element (IE), u , such that $a \circ u = a$.
- There is a unique inverse a^{-1} for every element such that $a \circ a^{-1} = u$ where $u = IE$.

Additional rule:

If the property of commutativity is applicable, i.e. $a \circ b = b \circ a$, then the group is an Abelian group.

Examination questions

Question 1

Four new exciting paint colours — red, blue, purple and clear — were released onto the market. When clear was mixed with any other colour, the result was clear.

Red mixed with blue went purple, red mixed with purple went blue, blue mixed with purple went red. Colours mixed with their same colour remain the same as the original.

Does this colour mixing form a group?
(Justify your answer.)

Question 2

The movements of a toy car are:

- (1) turn left ($\leftarrow$)
- (2) turn right ($\rightarrow$)
- (3) no change (i.e. remain stationary) (ϕ)
- (4) turn about 180° ($\odot$)

Construct a Cayley table and show that this set of movements forms a group.

Question 3

Construct a Cayley table for $[(\text{mod } 4), +]$ and determine if it is an Abelian group.

Question 4

Show that the functions $f(t) = t$; $g(t) = -t$, $h(t) = \frac{1}{t}$ and $k(t) = -\frac{1}{t}$ form a group under composition.

Question 5

Find the missing entries in the set $S\{p, q, r, s, t\}$ if the following table forms a group.

$\circ$	p	q	r	s	t
p	p	—	—	—	—
q	—	r	—	—	p
r	—	s	t	p	—
s	—	t	—	q	r
—	—	—	—	—	—

Question 6

Let $S = \{1, 3, 5, 15\}$ and let $\oplus$ and $\otimes$ be an operation on S such that, if a and b are elements of S , then:

- $a \oplus b =$ lowest common multiple of a and b
 $a \otimes b =$ highest common factor of a and b

- (i) Complete the following tables which show the results of applying $\oplus$ and $\otimes$ to every pair of elements of S . For example, the lowest common multiple of 3 and 1 is 3.

$$a \oplus b$$

$a \backslash b$	1	3	5	15
1				
3				
5				
15				

$$a \otimes b$$

$a \backslash b$	1	3	5	15
1				
3				
5				
15				

- (ii) Explain how the tables show that the operations $\oplus$ and $\otimes$ are commutative.
- (iii) Write down the value of u , an element of S , such that $a \oplus u = a$ for every element of a in S .
- (iv) Write down the value of r , an element of S , such that $a \otimes r = a$ for every element of a in S .
- (v) For every element a in S let $a' = \frac{15}{a}$.

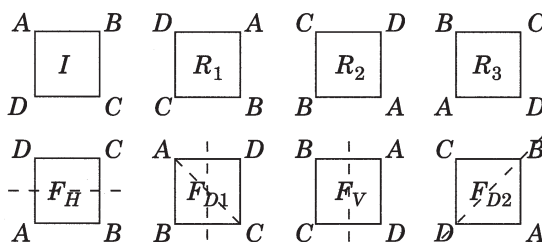
Complete the table below and discuss your results.

a	a'	$a \oplus a'$	$a \otimes a'$
1			
3			
5			
15			

- (vi) Complete the table below for every combination of a and b in S , and use the results to prove that $(a \oplus b)' = a' \otimes b'$.

a	b	$a \oplus b$	$(a \oplus b)'$	a'	b'	$a' \otimes b'$
1	1					
1	3					
1	5					
1	15					
3	1					
3	3					

Question 7



- (i) Does this set form a group?
- (ii) Is this set commutative?

WORKED SOLUTIONS

Question 1

Resultant table (using information given):

	Red	Blue	Clear	Purple
Red	Red	Purple	Clear	Blue
Blue	Purple	Blue	Clear	Red
Clear	Clear	Clear	Clear	Clear
Purple	Blue	Red	Clear	Purple

(a) Using the above table, the set of results is closed.
i.e. red, blue, clear, purple are all members of the original set

(b) Associativity: ($\circ$ = mix)
 $(\text{Red} \circ \text{Blue}) \circ \text{Purple} = (\text{Purple}) \circ \text{Purple} = \text{Purple}$
 $\text{Red} \circ (\text{Blue} \circ \text{Purple}) = \text{Red} \circ (\text{Red}) = \text{Red}$
 $\therefore$ This is *not* a group, as associativity is not true.

(Optional) (c) Identity: $\text{Red mix Red} = \text{Red}$
 $\text{Blue mix Blue} = \text{Blue}$
 $\text{Clear mix Clear} = \text{Clear}$
 $\text{Purple mix Purple} = \text{Purple}$
 No identity element exists ($\Rightarrow$ no group)

Question 2

Properties of group:

(a) Closed under 'o'

	$\leftarrow$	$\rightarrow$	ϕ	$\circlearrowleft$
$\leftarrow$	$\circlearrowleft$	ϕ	$\leftarrow$	$\rightarrow$
$\rightarrow$	ϕ	$\circlearrowleft$	$\rightarrow$	$\leftarrow$
ϕ	$\leftarrow$	$\rightarrow$	ϕ	$\circlearrowleft$
$\circlearrowleft$	$\rightarrow$	$\leftarrow$	$\circlearrowleft$	ϕ

(b) Associative: (i) $(\leftarrow \circ \rightarrow) \circ \phi = (\phi) \circ \phi = \phi$
 $\leftarrow \circ (\rightarrow \circ \phi) = \leftarrow \circ (\rightarrow) = \phi$
 (ii) $(\leftarrow \circ \rightarrow) \circ \circlearrowleft = (\phi) \circ \circlearrowleft = \circlearrowleft$
 $\leftarrow \circ (\rightarrow \circ \circlearrowleft) = \leftarrow \circ (\leftarrow) = \circlearrowleft$ etc.

(c) Identity element u (such that $a \circ u = a \cup a$):

$$\begin{aligned} \leftarrow \circ \phi &= \leftarrow \\ \rightarrow \circ \phi &= \rightarrow \\ \phi \circ \phi &= \phi \\ \circlearrowleft \circ \phi &= \circlearrowleft \end{aligned} \quad \text{i.e. } IE = \phi$$

(d) Unique inverse a^{-1} for every element such that $a \circ a^{-1} = IE = \phi$

$$\begin{aligned} \therefore \rightarrow \circ \leftarrow &= \phi \\ \leftarrow \circ \rightarrow &= \phi \\ \phi \circ \phi &= \phi \\ \circlearrowleft \circ \circlearrowleft &= \phi \end{aligned}$$

Therefore since all group conditions have been fulfilled,
the movements of the toy car form a group.

Question 3

$$S = \{0, 1, 2, 3\}$$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Properties of group:

(1) Closed under '+':

From the above table 0, 1, 2, 3, $\in S$, hence closed under '+'

(2) Associative: $[(0+1)+2 = (1+2) = 3] = [0+(1+2) = 0+3 = 3]$

$$[(0+1)+3 = 1+3 = 0] = [0+(1+3) = 0+0 = 0]$$

(3) Only one IE, u , such that $a+u = a \quad \forall a \in S$:

$$0+0 = 0$$

$$1+0 = 1$$

$$2+0 = 2$$

$$3+0 = 3 \quad \text{hence } u = 0$$

(4) Unique inverse a^{-1} for every element such that $a+a^{-1} = 0$

$$\left. \begin{array}{l} 0+0 = 0 \\ 1+3 = 0 \\ 2+2 = 0 \\ 3+1 = 0 \end{array} \right\} \Rightarrow \text{unique inverse}$$

(5) Abelian group if commutative: $0+1 = 1 = 1+0$

$$0+2 = 2 = 2+0$$

$$0+3 = 3 = 3+0$$

$$1+1 = 2 = 1+1$$

$$1+2 = 3 = 2+1$$

$$1+3 = 0 = 3+1$$

$$2+3 = 1 = 3+2$$

$$3+3 = 2 = 3+3$$

(i.e. symmetric about the diagonal)

Hence $[(\text{mod } 4), +]$ forms an Abelian group.

Question 4

$$f(t) = t, \quad g(t) = -t, \quad h(t) = \frac{1}{t}, \quad k(t) = -\frac{1}{t}$$

$\circ$	f	g	h	k
f	f	g	h	k
g	g	f	k	h
h	h	k	f	g
k	k	h	g	f

$$f[f(t)] = f(t) = t = f(t)$$

$$g[f(t)] = g(t) = -t = g(t)$$

$$f[g(t)] = f(-t) = -t = g(t)$$

$$g[g(t)] = g(-t) = -(-t) = t = f(t)$$

$$f[h(t)] = f\left(\frac{1}{t}\right) = \frac{1}{t} = h(t)$$

$$g[h(t)] = g\left(\frac{1}{t}\right) = -\frac{1}{t} = k(t)$$

$$f[k(t)] = f\left(-\frac{1}{t}\right) = -\frac{1}{t} = k(t)$$

$$g[k(t)] = g\left(-\frac{1}{t}\right) = -\left(-\frac{1}{t}\right) = \frac{1}{t} = h(t)$$

continued ...

$$h[f(t)] = h(t) = \frac{1}{t} = h(t)$$

$$k[f(t)] = k(t) = -\frac{1}{t} = k(t)$$

$$h[g(t)] = h(-t) = \frac{1}{-t} = k(t)$$

$$k[g(t)] = k(-t) = \frac{-1}{-t} = \frac{1}{t} = h(t)$$

$$h[h(t)] = h\left(\frac{1}{t}\right) = \frac{1}{\frac{1}{t}} = t = f(t)$$

$$k[h(t)] = k\left(\frac{1}{t}\right) = \frac{-1}{\frac{1}{t}} = -t = g(t)$$

$$h[k(t)] = h\left(-\frac{1}{t}\right) = \frac{1}{-\frac{1}{t}} = -t = g(t)$$

$$k[k(t)] = k\left(-\frac{1}{t}\right) = \frac{-1}{-\frac{1}{t}} = t = f(t)$$

Properties of groups:

(a) Closed under composition:

From the table, $\{f, g, h, k\} \in S$, hence closed.

(b) Associative:

$$(f \circ g) \circ h = (g) \circ h = k = f \circ (g \circ h) = f \circ (k) = k$$

$$(f \circ h) \circ g = (h) \circ g = k = f \circ (h \circ g) = f \circ (k) = k$$

$$(g \circ h) \circ k = (k) \circ k = f = g \circ (h \circ k) = g \circ (g) = f \quad \text{etc.}$$

(c) Identity element: $f \circ f = f$

$$g \circ f = g$$

$$h \circ f = h$$

$$k \circ f = k \quad \text{hence } IE = f$$

(d) Unique inverse a^{-1} for every element such that $a \circ a^{-1} = I = f$

$$f \circ f = f; \quad g \circ g = f; \quad h \circ h = f; \quad k \circ k = f$$

Hence each element has its own inverse,

$\therefore f, g, h, k$ form a group under composition.

Hence all group properties are satisfied,

$\therefore$ the composition of functions forms a group.

Question 5

$\circ$	p	q	r	s	t
p	p	(ii) q	(ii) r	(ii) s	(ii) t
q	(ii) q	r	(iv) s	(iv) t	p
r	(ii) r	s	t	p	(iv) q
s	(ii) s	t	(iii) p	q	r
(i) t	(ii) t	(iii) p	(iv) q	(iv) r	(iv) s

(i) This position must be t , since the group is closed.

(ii) A group must have an identity element. From the given information, the identity element must be p since $p \circ p = p$, $\therefore p = I$.

This information can be used to fill in:

$$q \circ p = q \quad r \circ p = r \quad s \circ p = s \quad t \circ p = t$$

$$p \circ q = q \quad p \circ r = r \quad p \circ s = s \quad p \circ t = t$$

(iii) In a group, each element must have an inverse such that

$$p \circ p^{-1} = I = p; \quad q \circ q^{-1} = I = p; \quad r \circ r^{-1} = I = p;$$

$$s \circ s^{-1} = I = p; \quad t \circ t^{-1} = I = p$$

Hence p must be in each row and each column.

(iv) Using associativity:

$$\begin{aligned}
 [s \circ (s \circ r) = s \circ p = s] &= [(s \circ s) \circ r = q \circ r = s] \\
 [s \circ (s \circ s) = s \circ q = t] &= [(s \circ s) \circ s = q \circ s = t] \\
 [q \circ (q \circ t) = q \circ p = q] &= [(q \circ q) \circ t = r \circ t = q] \\
 [r \circ (r \circ t) = r \circ q = s] &= [(r \circ r) \circ t = t \circ t = s] \\
 [q \circ (s \circ r) = q \circ p = q] &= [(q \circ s) \circ r = t \circ r = q] \\
 [q \circ (s \circ s) = q \circ q = r] &= [(q \circ s) \circ s = t \circ s = r]
 \end{aligned}$$

Question 6(i) $a \oplus b$ $\oplus = \text{LCM}$

$\backslash b$	1	3	5	15
1	1	3	5	15
3	3	3	15	15
5	5	15	5	15
15	15	15	15	15

 $a \otimes b$ $\otimes = \text{HCF}$

$\backslash b$	1	3	5	15
1	1	1	1	1
3	1	3	1	3
5	1	1	5	5
15	1	3	5	15

(ii)

$$\begin{aligned}
 1 \oplus 1 &= 1 = 1 \oplus 1 \\
 1 \oplus 3 &= 3 = 3 \oplus 1 \\
 1 \oplus 5 &= 5 = 5 \oplus 1 \\
 1 \oplus 15 &= 15 = 15 \oplus 1 \\
 3 \oplus 3 &= 3 = 3 \oplus 3 \\
 3 \oplus 5 &= 15 = 5 \oplus 3 \\
 3 \oplus 15 &= 15 = 15 \oplus 3 \\
 5 \oplus 5 &= 5 = 5 \oplus 5 \\
 5 \oplus 15 &= 15 = 15 \oplus 5
 \end{aligned}$$

Hence $\oplus$ is commutative.

$$\begin{aligned}
 1 \otimes 1 &= 1 = 1 \otimes 1 \\
 1 \otimes 3 &= 1 = 3 \otimes 1 \\
 1 \otimes 5 &= 1 = 5 \otimes 1 \\
 1 \otimes 15 &= 1 = 15 \otimes 1 \\
 3 \otimes 3 &= 3 = 3 \otimes 3 \\
 3 \otimes 5 &= 1 = 5 \otimes 3 \\
 3 \otimes 15 &= 3 = 15 \otimes 3 \\
 5 \otimes 5 &= 5 = 5 \otimes 5 \\
 5 \otimes 15 &= 5 = 15 \otimes 5
 \end{aligned}$$

Hence $\otimes$ is commutative.

(iii)

$$\begin{aligned}
 1 \oplus 1 &= 1 \\
 3 \oplus 1 &= 3 \\
 5 \oplus 1 &= 5 \\
 15 \oplus 1 &= 15
 \end{aligned}$$

Hence $1E = u = 1$

(iv)

$$\begin{aligned}
 1 \otimes 15 &= 1 \\
 3 \otimes 15 &= 3 \\
 5 \otimes 15 &= 5 \\
 15 \otimes 15 &= 15
 \end{aligned}$$

Hence $r = 15$

(v) $a' = \frac{15}{a}$

a	a'	$a \oplus a'$	$a \otimes a'$
1	15	15	1
3	5	15	1
5	3	15	1
15	1	15	1

Results show that $a \otimes a' = 1 \equiv u$
and $a \oplus a' = 15 = r$

(vi)

			$= \frac{15}{a \oplus b}$	$= \frac{15}{a}$	$= \frac{15}{b}$	
a	b	$a \oplus b$	$(a \oplus b)'$	a'	b'	$a' \otimes b'$
1	1	1	15	15	15	15
1	3	3	5	15	5	5
1	5	5	3	15	3	3
1	15	15	1	15	1	1
3	1	3	5	5	15	5
3	3	3	5	5	5	5
3	5	15	1	5	3	1
3	15	15	1	5	1	1
5	1	5	3	3	15	3
5	3	15	1	3	5	1
5	5	5	3	3	3	3
5	15	15	1	3	1	1
15	1	15	1	1	15	1
15	3	15	1	1	5	1
15	5	15	1	1	3	1
15	15	15	1	1	1	1

Hence using columns $(a \oplus b)'$ and $a' \otimes b'$, we see they are equivalent.

Question 7

(i)

$\circ$	I	R_1	R_2	R_3	F_H	F_{D1}	F_V	F_{D2}
I	I	R_1	R_2	R_3	F_H	F_{D1}	F_V	F_{D2}
R_1	R_1	R_2	R_3	I	F_{D2}	F_H	F_{D1}	F_V
R_2	R_2	R_3	I	R_1	F_V	F_{D2}	F_H	F_{D1}
R_3	R_3	I	R_1	R_2	F_{D1}	F_V	F_{D2}	F_H
F_H	F_H	F_{D1}	F_V	F_{D2}	I	R_1	R_2	R_3
F_{D1}	F_{D1}	F_V	F_{D2}	F_H	R_1	I	R_3	R_2
F_V	F_V	F_{D2}	F_H	F_{D1}	R_2	R_3	I	R_1
F_{D2}	F_{D2}	F_H	F_{D1}	F_V	R_3	R_2	R_1	I

1. Does $a \circ b = c$ where $a, b, c \in S$?

All operations produced elements of the set, therefore it is closed.

2. Does $(a \circ b) \circ c = a \circ (b \circ c) \quad \forall \quad a, b, c \in S$?

$$(R_2 \circ F_H) \circ F_{D1} = F_V \circ F_{D1} = R_3 = R_2 \circ (F_H \circ F_{D1}) \\ = R_2 \circ R_1 = R_3$$

$$(F_V \circ F_{D2}) \circ R_3 = R_1 \circ R_3 = I = F_V \circ (F_{D2} \circ R_3) \\ = F_V \circ F_V = I \quad \text{etc.}$$

$\therefore$ Associativity exists.

3. Does $\exists I$ where $a \circ I = a \quad \forall \quad a \in S$?

$$\begin{array}{ll} I \circ I = I & F_H \circ I = F_H \\ R_1 \circ I = R_1 & F_{D1} \circ I = F_{D1} \\ R_2 \circ I = R_2 & F_V \circ I = F_V \\ R_3 \circ I = R_3 & F_{D2} \circ I = F_{D2} \end{array}$$

$\therefore$ Identity element = I

4. Does $\exists a^{-1}$ such that $a \circ a^{-1} = I \quad \forall a \in S$?

$$\begin{aligned} I \circ I &= I \\ R_1 \circ R_3 &= I \\ R_2 \circ R_2 &= I \\ R_3 \circ R_1 &= I \\ F_H \circ F_H &= I \\ F_{D1} \circ F_{D1} &= I \\ F_V \circ F_V &= I \\ F_{D2} \circ F_{D2} &= I \end{aligned}$$

$\therefore$ each a has a^{-1} such that $a \circ a^{-1} = I$.

Since the set satisfies all 4 conditions, it is a group.

(b) Does $a \circ b = b \circ a \quad \forall a, b \in S$?

$$\begin{aligned} R_3 \circ F_H &= F_{D1} \\ F_H \circ R_3 &= F_{D2} \end{aligned}$$

$\therefore$ the group is not commutative (i.e. not Abelian group).